TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 222, 1976

THE MODULE OF INDECOMPOSABLES FOR FINITE H-SPACES[1]

BY

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ABSTRACT. The module of indecomposables obtained from the mod p cohomology of a finite H-space is studied for p odd. General structure theorems are obtained, first, regarding the possible even dimensions in which this module can be nonzero and, secondly, regarding how the Steenrod algebra acts on the module.

1. **Introduction.** An *H-space* (X, μ) is a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map $\mu: X \times X \longrightarrow X$ with two-sided homotopy unit. An H-space (X, μ) will be said to be *mod p finite* if $H^*(X; Z_p)$ is a finite-dimensional Z_p module. (Z_p are the integers reduced mod p.) If (X, μ) is a mod p finite H-space then a knowledge of the module of indecomposables $Q(H^*(X; Z_p))$ is important and useful for several reasons. It would be a major step in determining the possible Hopf algebra structures on $H^*(X; Z_p)$. Moreover, as the results of [2] and [5] indicate, a knowledge of $Q(H^*(X; Z_p))$ and, in particular, of $Q(H^{\mathrm{even}}(X; Z_p))$, is necessary for any systematic understanding of the occurrence of p torsion in $H^*(X; Z)$. In this paper we will study the structure of $Q(H^{\mathrm{even}}(X; Z_p))$ when p is odd.

Given a positive integer m let $m = \Sigma m_s p^s$ be its p-adic expansion. We say m is *binary* (with respect to p) if $m_s = 0$ or 1 for each s.

THEOREM 1:1. *Let p be odd and let (X, μ) be a* mod *p finite H-space such that $H^*(X; Z_p)$ is a coassociative Hopf algebra. Then $Q(H^{2m}(X; Z_p)) = 0$ unless m is binary.*

We will speak of $Q(H^{\mathrm{even}}(X; Z_p))$ as being binary if the condition in 1:1 is satisfied.

Now $Q(H^*(X; Z_p))$ is a module over the Steenrod algebra $A^*(p)$. If X is

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Received by the editors March 19, 1975.

AMS (MOS) subject classifications (1970). Primary 57F25; Secondary 55G20.

Key words and phrases. H-space, mod p finite, Hopf algebra, indecomposable, primitive, Steenrod algebra, secondary operation.

(1) The work for this paper was completed while the author was visiting at the Massachusetts Institute of Technology.

mod p finite then, by [1], $\beta_p Q(H^{\text{even}}(X; Z_p)) = 0$. Hence, $Q(H^{\text{even}}(X; Z_p))$ is a Steenrod submodule of $Q(H^*(X; Z_p))$. Theorem 1:1 has consequences for this Steenrod module structure.

We first define some functions. Let $\gamma(0) = 0$ and, for $s \geq 1$, let $\gamma(s) = \Sigma_{i=0}^{s-1} p^i$. Suppose m is binary and $m \equiv \gamma(s) \bmod p^s$ but $m \not\equiv \gamma(s+1) \bmod p^{s+1}$. Then, let $\rho(m) = \gamma(s)$ and $\delta(m) = (m - \gamma(s))/p$. Given m with p-adic expansion $\Sigma m_s p^s$ let $\omega(m) = \Sigma_s m_s$.

THEOREM 1:2. *Let p be odd and let (X, μ) be a* mod p *finite H-space such that $H^*(X; Z_p)$ is a coassociative Hopf algebra. Then the function ω defines a splitting of $Q(H^{\text{even}}(X; Z_p))$ as a Steenrod module, namely $Q(H^{\text{even}}(X; Z_p)) = \bigoplus_{s \geq 1} M_s$ where $M_s = \Sigma_{\omega(m)=s} Q(H^{2m}(X; Z_p))$.*

THEOREM 1:3. *Let p be odd and let (X, μ)* be a mod p *finite H-space such that $H^*(X; Z_p)$ is a coassociative Hopf algebra. Then, for m binary*

$$Q(H^{2m}(X; Z_p)) = P^{\delta(m)} Q(H^{2\delta(m)+2\rho(m)}(X; Z_p)).$$

In particular $Q(H^{2m}(X; Z_p)) = 0$ unless $m \equiv 1 \bmod p$.

By using the results of [2] we can also deduce from 1:3:

THEOREM 1:4. *Let p be odd and let (X, μ) be a* mod p *finite H-space such that $H^*(X; Z_p)$ is a coassociative Hopf algebra. Then, for $m \geq 1$, the rank of $Q(H^{2m}(X; Z_p))$ as a Z_p-module is bounded by the rank of $Q(H^{2m-1}(X; Z_p))$ as a Z_p-module.*

Results 1:1, 1:2 and 1:3 are motivated by the result of Zabrodsky [10] that $Q(H^{\text{even}}(X; Z_p))$ is generated as a Steenrod module by $\Sigma_{s \geq 1} Q(H^{2\gamma(s)}(X; Z_p))$. Our approach, while based on [10], is somewhat different. The principal tool used is that of secondary operations. We first generalize the construction of secondary operations in the cohomology of H-spaces as given in [9]. We then apply these secondary operations to study $Q(H^{\text{even}}(X; Z_p))$. However, as compared to [10], our approach places more emphasis on algebra and less on homotopy theory. This will make the paper relatively self-contained.

In §2 we will do some Hopf algebra preliminaries and, in particular, study the Hopf algebra structure of $H^*(X; Z_p)$ and $H_*(X; Z_p)$. In §3 we will describe the method of constructing secondary operations for the cohomology of H-spaces. In §4 we will study the structure of $Q(H^{\text{even}}(X; Z_p))$ and, in particular, prove 1:2, 1:3, 1:4, at least assuming 1:1 is true. In §5 we will prove 1:1.

2. **Hopf algebras.** The basic reference for Hopf algebras is [8]. We will use the term "Hopf algebra" in the sense of what is there called a

"quasi-Hopf algebra". We will assume that all Hopf algebras are over Z_p and are graded, connected and of finite type. Given a graded module A we use the symbol $\overline{A}$ to denote the elements of positive dimension. Let (X, μ) be an H-space. Then $H^* = H^*(X; Z_p)$ and $H_*(X; Z_p)$ have natural structures as Hopf algebras over $A^*(p)$ induced by μ and the diagonal map $\Delta: X \longrightarrow X \times X$. The action of $A^*(p)$ on H_* is a right one and is obtained by duality from the usual left action of $A^*(p)$ on H^*. All Hopf algebras will be assumed to be either associative and commutative like H^* or coassociative and cocommutative like H_*.

Given a Hopf algebra A we define its dual A^* by the rule

$$(A^*)^n = \mathrm{Hom}(A^n; Z_p).$$

We use $P(A)$ and $Q(A)$ to indicate primitives and indecomposables respectively. We observe that $P(A)$ and $Q(A^*)$ are dual in the sense of a submodule of A being dual to a quotient module of A^*. In particular H^* and H_* are dual Hopf algebras and, in that case, the Steenrod module structures of $Q(H^*)$ and $P(H_*)$ are dual as well.

(2:1) The natural map $P(A) \longrightarrow Q(A)$ is injective in odd dimension if A is commutative and associative. It is surjective in odd dimension if A is cocommutative and coassociative.

If B is a normal sub Hopf algebra of A then the quotient Hopf algebra $A//B = A/\overline{B}A$ is defined. Where there is no confusion we will use the same symbol for an element in A and for its image in a quotient module. We will do likewise for induced maps between quotient modules.

Given a Hopf algebra with comultiplication ψ we will use the symbol $\overline{\psi}$ to denote the reduced comultiplication defined by the rule

$$\overline{\psi}(x) = \psi(x) - x \otimes 1 - 1 \otimes x \quad \text{for any } x \in A.$$

Given $x \in A$ we let $|x|$ denote the dimension of x. Given an algebra A we define the Lie bracket product [,] by the rule

$$[x, y] = xy - (-1)^{|x|\,|y|}yx \quad \text{for any } x, y \in A.$$

In the rest of this section we will study the Hopf algebra structure of H^* when (X, μ) is an H-space as in 1:1. Our principal result is

THEOREM 2:2. *Let (X, μ) be a* mod p *finite H-space such that H^* is a coassociative Hopf algebra. Then H^* contains a sub Hopf algebra Γ over $A^*(p)$ such that*

(i) *the natural map $Q(\Gamma) \longrightarrow Q(H^*)$ is surjective in even dimensions,*
(ii) $\Gamma^{2j+1} = 0$ *for all j.*

The rest of this section will be devoted to proving 2:2. First, recall the following facts obtained by Browder using the Bockstein spectral sequence.

LEMMA 2:3. *If* $u, v \in P(H_{\mathrm{odd}})$ *then* $u^2 = v^2 = 0$, $u\beta_p = v\beta_p = 0$, *and* $uv = -vu$.

See [6] for proofs. Let Λ be the sub Hopf algebra of H_* generated over $A^*(p)$ by $P(H_{\mathrm{odd}})$. By 2:3 it is the exterior Hopf algebra generated by $P(H_{\mathrm{odd}})$. Suppose for the moment that the following is true.

LEMMA 2:4. Λ *is a normal sub Hopf algebra of* H_*.

Then $\Omega = H_*//\Lambda$ is defined as a Hopf algebra over $A^*(p)$. We let $\Gamma = \Omega^* \subset H^*$. Then Γ satisfies (i) and (ii) of 2:2.

PROPERTY (i). Since $H^*//\overline{\Gamma} = \Lambda^*$ there is an exact sequence

$$(*) \qquad Q(\Gamma) \longrightarrow Q(H^*) \longrightarrow Q(\Lambda^*) \longrightarrow 0$$

(see 3:11 of [8]). But, by construction, $P_{\mathrm{even}}(\Lambda) = 0$ and so $Q^{\mathrm{even}}(\Lambda^*) = 0$.

PROPERTY (ii). It suffices to show $Q^{\mathrm{odd}}(\Gamma) = 0$. We will do this by induction. We have that $Q^{-1}(\Gamma) = 0$. So assume by induction that $Q^{2i+1}(\Gamma) = 0$ if $i < k$. To show that $Q^{2k+1}(\Gamma) = 0$ it suffices to show that the natural map $Q^{2k+1}(\Gamma) \longrightarrow Q(H^{2k+1})$ is: (a) trivial, and, (b) a monomorphism. Fact (a) follows from the exact sequence $(*)$ since, by construction $P_{\mathrm{odd}}(\Lambda) \cong P(H_{\mathrm{odd}})$ and so $Q(H^{\mathrm{odd}}) \cong Q^{\mathrm{odd}}(\Lambda^*)$. Fact (b) follows from the commutative diagram

$$\begin{array}{ccc} P^{2k+1}(\Gamma) & \longrightarrow & Q^{2k+1}(\Gamma) \\ \downarrow & & \downarrow \\ P(H^{2k+1}) & \longrightarrow & Q(H^{2k+1}). \end{array}$$

For, by 2:1 and the induction hypothesis, the top vertical map is an isomorphism and the bottom vertical map is a monomorphism. Further, the left vertical map is an obvious monomorphism.

Thus to prove 2:2 we are left with proving 2:4. We first recall some facts about normal sub Hopf algebras. Let A be an associative, coassociative, cocommutative Hopf algebra. Let B be a sub Hopf algebra of A and C a sub Hopf algebra of B. Suppose C is normal in both A and B.

LEMMA 2:5. $B//C$ *is a sub Hopf algebra of* $A//C$, *that is, the induced map* $B//C \longrightarrow A//C$ *is a monomorphism.*

LEMMA 2:6. $B//C$ *is normal in* $A//C$ *if, and only if,* B *is normal in* A.

LEMMA 2:7. *If* B *is normal in* A *then* $A//B$ *and* $(A//C)//(B//C)$ *are isomorphic as Hopf algebras.*

LEMMA 2:8. *If B is normal in A then the sequence* $0 \longrightarrow P(B) \longrightarrow P(A) \longrightarrow P(A//B) \longrightarrow 0$ *is exact in odd dimensions.*

Lemma 2:5 follows from the fact (see 4:4 of [8]) that we can form a commutative diagram of C-modules:

$$\begin{array}{ccc} B & \cong & C \otimes B//C \\ \downarrow & & \downarrow \\ A & \cong & C \otimes A//C \end{array}$$

Lemma 2:6 is an exercise in working with cosets. Regarding Lemma 2:7 we have a surjective map of Hopf algebras $(A//C)//(B//C) \longrightarrow A//B$ induced from the map $A//C \longrightarrow A//B$. The following identities (see 4:4 of [8]) then show the map must be an isomorphism:

$$B \otimes A//B \cong A \cong C \otimes A//C \cong C \otimes B//C \otimes A//C//B//C \cong B \otimes A//C//B//C.$$

For Lemma 2:8 we need only show $P(A) \longrightarrow P(A//B)$ is surjective in odd dimensions. The rest follows by an argument similar to that given in the proof of 2:2(ii). If we dualize it suffices to show $Q(D) \longrightarrow Q(A^*)$ is injective where D is a sub Hopf algebra of A^*. Given $x \in Q(D^{2n+1})$ represented by $x \in D^{2n+1}$ let D' be the sub Hopf algebra of D generated by the elements of dimension $2n$ or less. Then x is nonzero and primitive in $D//D'$. As in 2:5 x is nonzero and primitive in $A^*//D'$. By 4:21 of [8] x is nondecomposable in $A^*//D'$ and thus in A^* as well.

Now let $\Lambda(n)$ be the sub Hopf algebra generated by $\Sigma_{s \geq n} P(H_{2s+1})$. To prove 2:4 we will show by decreasing induction on n that $\Lambda(n)$ is normal in H_*. Now if we pick n large then $\Lambda(n)$ is trivial since H_* is finite. Assume, by induction, that $\Lambda(n+1)$ is normal in H_*. Let $\Omega(n+1) = H_*//\Lambda(n+1)$. By 2:3 $\Lambda(n+1)$ is normal in $\Lambda(n)$. Let $\Phi(n+1) = \Lambda(n)//\Lambda(n+1)$. By 2:5 $\Phi(n+1)$ is a sub Hopf algebra of $\Omega(n+1)$. By 2:6, in order to show $\Lambda(n)$ is normal in H_*, it suffices to show $\Phi(n+1)$ is normal in $\Omega(n+1)$. We will show more, namely,

LEMMA 2:9. $\Phi(n+1)$ *is central in* $\Omega(n+1)$ *i.e.* $[x, y] = 0$ *for all* $x \in \Phi(n+1)$, $y \in \Omega(n+1)$ *where* $[\ ,\ ]$ *is the Lie bracket product.*

PROOF. Since $\Phi(n+1)$ is a primitively generated exterior Hopf algebra on generators of dimension $2n+1$, we can assume x is such a generator. We can assume y is indecomposable. If y is odd dimensional then by 2:1 y can be assumed to be primitive and hence, by 2:3 and 2:8, $[x, y] = 0$. If y is even dimensional then by an (increasing) induction on the dimension of $\Omega(n+1)$ we can assume $[x, y]$ is primitive. For, if $\Delta_*(y) = \Sigma_s y'_s \otimes y''_s$, then

$$\Delta_*[x, y] = \sum_s [x, y'_s] \otimes y''_s \pm y'_s \otimes [x, y''_s].$$

By 2:8, $\Omega(n + 1)$ has no nonzero primitive elements of odd dimension greater than $2n + 1$. Hence $[x, y] = 0$.

3. **Secondary operations.** Secondary operations are associated to Adem relations and are constructed by means of universal example (E, u, v) (see §1 of [10]). For our purposes we are interested in unstable Adem relations. By an unstable relation in dimension s we mean any relation which holds among the elements of $A^*(p)$ as they act on the fundamental class $\iota_s \in H^s(K(Z_p, s); Z_p)$. Any such relation is obtained from an ordinary stable Adem relation by equating to zero all Steenrod operations which act trivially on ι_s. An element $\theta \in A^*(p)$ is said to be of excess s or less ($e(\theta) \leqslant s$) if $\theta(\iota_s) \neq 0$.

Now suppose $m > k \geqslant 0$. Let $\theta \in A^k(p)$ where $e(\theta) \leqslant 2m - k$. Let

(3:1) $$\sum_s a_s b_s = \beta_p P^m \theta$$

be an unstable Adem relation in dimension $2m - k + 1$ such that, for each s, $|b_s| > |\theta| = k$ or $e(b_s) < e(\theta)$. It gives rise to the unstable Adem relation

(3:2) $$\sum a_s b_s = 0$$

in dimension $2m - k$. We will define, by means of universal example (E, u, v) an unstable secondary operation ϕ associated with 3:2 such that

PROPOSITION 3:3. *If $\phi(x)$ and $\phi(y)$ are defined then*

$$\phi(x + y) = \phi(x) + \phi(y) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\theta(x)^i \otimes \theta(y)^{p-i}.$$

Equivalently, if α: $E \times E \longrightarrow E$ is the loop multiplication defined on E, then

$$\bar{\alpha}^*(v) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}\theta(u)^i \otimes \theta(u)^{p-i}.$$

For a proof of 3:3 see [3] and [4].

Because of the manner in which 3:3 is obtained we will violate the usual terminology and speak of ϕ as the unstable secondary operation associated with 3:1 (rather than 3:2).

From 3:3 we obtain our basic result about secondary operations.

THEOREM 3:4. *Let (X, μ) be an H-space. Let ϕ be the unstable operation associated with* 3:1. *Let $B \subset H^*$ be an $A^*(p)$ module and let $I(B)$ be the ideal* *generated by* $\overline{B}$. *Given* $x \in H^{2m-k}$, $y \in H^{2m}$ *such that* $y = \theta(x)$, $x \in \bigcap_s \ker b_s$ *and* $\bar{\mu}^*(x) \in I(B) \otimes I(B)$ *then, in* $H^*//B \otimes H^*//B$,

$$\bar{\mu}^*\phi(x) \equiv \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} y^i \otimes y^{p-i} \bmod \sum_s \operatorname{im} a_s.$$

REMARK. This theorem generalizes 3:2 of [9]. Those cases are obtained by letting $x = y$ and $\theta = 1 \in A^0(p)$. However applications of 3:2 of [9] cannot be generalized so easily. There is a difficulty encountered when we attempt to apply our new operations. For, assume x and y are nondecomposable and that ϕ is defined on x. This will be the usual situation. Then we can find B such that $\bar{\mu}^*(x) \in I(B) \otimes I(B)$. If $x \neq y$, the difficulty is to find B such that $y \notin B$ as well. We overcome this difficulty when proving Proposition 5:3 by assuming that H^* is coassociative.

Theorem 3:4 is obtained from 3:3 in the same way as Theorem 3:2 of [9] is obtained from Proposition 3:1 of [9]. Our restrictions on the b_s element ensure that $\theta(u) \neq 0$.

4. **The $A^*(p)$ structure of $Q(H^*)$.** For this section assume that p is odd and that (X, μ) is a mod p finite H-space such that H^* is a coassociative Hopf algebra. We will study the structure of $Q(H^{\text{even}})$ and, in particular, prove 1:2, 1:3 and 1:4, at least modulo the proof of 1:1.

We begin with some results obtained by using the secondary operations discussed in the last section. We will state the results in terms of $P(H_*)$ since they are clearer in that form. Let $\{Q_s\}_{s \geq 0}$ be the Milnor elements of $A^*(p)$ (see [7]). In particular $Q_0 = \beta_p$. Given s, $m \geq 0$ we have the relation

(4:1) $$Q_s P^m = P^m Q_s - Q_{s+1} P^{m-p^s}.$$

We can deduce from 4:1 the relation

(4:2) $$Q_0 P^m = \sum_s (-1)^s P^{m-\gamma(s)} Q_s.$$

(In both 4:1 and 4:2 and in any later case we use the convention that $P^n = 0$ if n is negative.)

PROPOSITION 4:3. *Given* $0 \neq y \in P(H_{2m})$ *where* $m \neq \gamma(s)$ *for any* $s \geq 1$ *and* $\bigotimes_{i=1}^p y \in \bigcap_{s \geq 1} \ker P^{m-\gamma(s)}$, *then* $y^p \neq 0$. (*Here* $\bigotimes_{i=1}^p y$ *is an element of* $\bigotimes_{i=1}^p H_*$ *and the action of* $A^*(p)$ *on* $\bigotimes_{i=1}^p H_*$ *is the one obtained by the Cartan formula from the action of* $A^*(p)$ *on* H_*.)

PROOF OF 4:3. For $k \geq 1$ define a map $u^k\colon H_1^* \longrightarrow \bigotimes_{i=1}^k H^*$ by the recursive formula that $u^1 =$ the identity and $u^k = (u^{k-1} \otimes 1)\bar{u}^*$ where $\bar{u}^*$ is the reduced comultiplication. Expand y to a basis of $P(H_{2m})$ and pick $x \in Q(H^{2m})$ dual to y. Let Γ be as in 2:2. Let F_q be the sub Hopf algebra of Γ over $A^*(p)$ generated by $\Sigma_{i \leq q} \Gamma^{2i}$. By (ii) of 2:2 we can pick q such that $x \in Q(F_q)$, $x \notin Q(F_{q-1})$. Also x is represented by an element $x \in \Gamma^{2m}$ and we can choose it such that $x \in F_q$. Then $\bar{u}^*(x) \in F_{q-1} \otimes F_{q-1}$. By (i) of 2:2 $Q_s(x) = 0$ for $s \geqslant 0$. Hence the unstable secondary operation ϕ corresponding to 4:2 (obtained by letting $a_s = (-1)^s P^{m-\gamma(s)}$, $b_s = Q_s$ for each s) is defined on x and, in $H^*//F_{q-1} \otimes H^*//F_{q-1}$,

$$\bar{u}^*\phi(x) \equiv \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^{p-i} \otimes x^i \bmod \sum_{s \geqslant 0} \text{Image } P^{m-\gamma(s)}.$$

Hence, in $\bigotimes_{i=1}^{p} Q(H^*//F_{q-1})$,

$$u^p \phi(x) \equiv \bigotimes_{i=1}^{p} x \bmod \sum_{s \geqslant 1} \text{Image } P^{m-\gamma(s)}.$$

(Here we have mapped into $\bigotimes_{i=1}^{p} H^*//F_q$ using μ^p and then projected onto $\bigotimes_{i=1}^{p} Q(H^*//F_q)$ in the obvious manner. We can ignore Image P^m since x is indecomposable while P^m can be equated with the pth power map because of the dimensions involved). By the duality between x and y, 4:3 follows. Q.E.D.

LEMMA 4:4. *Given $x \neq y \in P(H_{2m})$; if $m \neq \gamma(s)$ for some $s \geqslant 1$ then $yP^q \neq 0$ for some $q \geqslant 1$. In particular, $yP^1 = 0$ if, and only if, $m \equiv 1 \bmod p$.*

PROOF. The first statement follows from 4:3 by a (decreasing) inductive argument on the dimension of y (or see [10]). Regarding the second statement, if $m \not\equiv 1 \bmod p$ then $m - \gamma(s) \not\equiv 0 \bmod p$ for any $s \geqslant 1$. Hence $P^1 P^{m-\gamma(s)-1} = \alpha P^{m-\gamma(s)}$ ($\alpha \in Z_p$) and $yP^1 \neq 0$ by 4:3. It then follows from the Adem relation $(P^1)^p = 0$ that $yP^1 = 0$ if $m \equiv 1 \bmod p$. Q.E.D.

The rest of this section is devoted to showing that if 1:1 is true then 1:2, 1:3, 1:4 are true. The proofs of 1:2 and 1:4 are straightforward.

(I) PROOF OF 1:2. The following is quite obvious.

LEMMA 4:5. *Let $Q(H^{\text{even}})$ be binary. Given $x \in Q(H^{2n})$, $y \in Q(H^{2m})$ such that $y = P^p(x) \neq 0$ then, in the p-adic expansions $n = \Sigma n_s p^s$, $m = \Sigma m_s p^s$, we must have $m_{k+1} = 1, m_k = 0, n_{k+1} = 0, n_k = 1$.*

Hence P^{p^k} merely "permutes" 0 and 1 and $\omega(n) = \omega(m)$. Since the elements $\{P^{p^k}\} \geqslant 0$ together with β_p generate $A^*(p)$, 1:2 follows from 1:1 and 4:5. Q.E.D.

(II) PROOF OF 1:4. This is merely a refinement of the proof of Theorem 4:7 of [2]. For a justification of all the statements of our proof we refer to [2].

Let $\{B_r d_r\}_{r \geqslant 1}$ be the mod p cohomology Bockstein spectral sequence for X. In particular $B_1 = H^*$. Given $x \in Q(H^{2m}) = Q(B_1^{2m})$ we can find $r \geqslant 1$ such that x survives to B_r and $d_r(y) = x$ for $y \in Q(B_r^{2m-1})$. If $m \not\equiv 0 \bmod p$ then every element of $Q(B_r^{2m-1})$ is represented in B_r^{2m-1} by a nondecomposable element $B_1^{2m-1} = H^{2m-1}$ which has survived to B_r^{2m-1}. But, if 1:3 is true then $m \equiv 1 \bmod p$. Hence, modulo the proof of 1:3, we are done. Q.E.D.

Before proving 1:3 we obtain some Steenrod module structure theorems for $Q(H^{\text{even}})$. Since we will have constant need of them we explicitly state the following Adem relations.

For $a, b \geqslant 0$ and p odd,

$$(4:6)\quad P^a P^b = \sum_{s \leqslant (a/p)} \alpha_s P^{a+b-s} P^s \quad \text{where } \alpha_s = (-1)^{a+s} \binom{(p-1)(b-s)-1}{a-ps}.$$

We also note the following consequence of 4:5:

LEMMA 4:7. *If $Q(H^{\text{even}})$ is binary then, for $s \geqslant 0$, $(P^{p^s})^2 = 0$ on $Q(H^{\text{even}})$.*
The main technical result which we need is

PROPOSITION 4:8. *Let $Q(H^{\text{even}})$ be binary. Let $K \geqslant 1$. If $m \equiv 0 \bmod p^K$ then $P^{m-q} P^q = 0$ on $Q(H^{\text{even}})$ unless $q \equiv 0 \bmod p^K$.*

PROOF. By induction on K.

$K = 1$. Suppose $q \not\equiv 0 \bmod p$. Hence $m - q \not\equiv 0 \bmod p$. We wish to show $P^{m-q} P^q = 0$ on $Q(H^{\text{even}})$. Let

$$q = p\bar{a} + a, \qquad m - q = p\bar{b} + b,$$

where $a \neq 0, b \neq 0$ and $a + b = p$. From 4:6 we deduce

$$P^q = \alpha (P^1)^a P^{p\bar{a}}, \qquad P^{m-q} = \beta (P^1)^b P^{p\bar{b}},$$

where $\alpha, \beta \in Z_p$. But either a or $b > 1$. Hence, by 4:7, either $P^q = 0$ or $P^{m-q} = 0$ on $Q(H^{\text{even}})$.

Now suppose the lemma is true for $i < K$ and consider the

General case K. Suppose $q \not\equiv 0 \bmod p^K$. Hence $m - q \not\equiv 0 \bmod p^K$. We again wish to show $P^{m-q} P^q = 0$ on $Q(H^{\text{even}})$. By the induction hypothesis we can assume $q \equiv m - q \equiv 0 \bmod p^{K-1}$. Let

$$q = p^K \bar{a} + p^{K-1} a, \qquad m - q = p^K \bar{b} + p^{K-1} b,$$

where $a \neq 0, b \neq 0$ and $a + b = p$. From the induction hypothesis and 4:6 we can deduce that, in $Q(H^{\text{even}})$,

$$P^q = \alpha (P^{p^{K-1}})^a P^{p^K \bar{a}}, \qquad P^{m-q} = \beta (P^{p^{K-1}})^b P^{p^K \bar{b}},$$

where $\alpha, \beta \in Z_p$. Again, since either a or $b > 1$ it follows, from 4:7, that $\mathcal{P}^{m-q}\mathcal{P}^q = 0$ on $Q(H^{\text{even}})$. Q.E.D.

Proposition 4:8 has one immediate corollary. For, from 4:8 and 4:6, it follows that, if $m = \Sigma m_s p^s$ is the p-adic expansion of m, then

$$\mathcal{P}^m = \alpha(\mathcal{P}^1)^{m_0}(\mathcal{P}^p)^{m_1} \cdots (\mathcal{P}^{p^t})^{m_t}$$

where $\alpha \in Z_p$. By 4:7 we can conclude

PROPOSITION 4:9. *$\mathcal{P}^m$ acts trivially on $Q(H^{\text{even}})$ unless m is binary. If $m = \Sigma p^{s_i}$ $(s_1 < s_2 < \cdots < s_t)$ then*

$$\mathcal{P}^m = \alpha \mathcal{P}^{p^{s_1}}\mathcal{P}^{p^{s_2}} \cdots \mathcal{P}^{p^{s_t}} \quad \text{on } Q(H^{\text{even}}) \qquad (\alpha \in Z_p).$$

(III) PROOF OF 1:3. We prove 1:3 by increasing induction on dimension. For $m = 1$ the result is trivial. Assume 1:3 is true in dimensions less than $2m$ and that $Q(H^{2m}) \neq 0$. Since we are assuming 1:1 is true m is binary.

LEMMA 4:10. $\rho(m) > 0$.

PROOF. By contradiction. Suppose $m \equiv 0 \bmod p$. By 4:4 $Q(H^{2m}) = \mathcal{P}^1 Q(H^{2n})$ where $n = m - p + 1$. By 4:5 $n \equiv 1 \bmod p^2$. By the induction hypothesis $Q(H^{2n}) = \mathcal{P}^{\delta(n)}Q(H^{2\delta(n)+2})$. Then $Q(H^{2m}) = \mathcal{P}^1\mathcal{P}^{\delta(n)}Q(H^{2\delta(n)+2})$ $= 0$ since, in dimension $2\delta(n) + 2$, $\mathcal{P}^{\delta(n)+1}$ agrees with the pth power map. Q.E.D.

LEMMA 4:11. *Given $y \in P(H_{2m})$ then $y^p = 0$.*

PROOF. By 4:10 $m \equiv 1 \bmod p$. By 4:4 $y \in \ker \mathcal{P}^1$. Hence $y^p \in P(H_{2pm})$ $\cap \ker \mathcal{P}^1$. By 4:4 $y^p = 0$. Q.E.D.

Now pick $0 \neq y \in P(H_{2m})$. By 4:3 and 4:11 $\bigotimes_{i=1}^p y \notin \bigcap_{s \geq 1} \ker \mathcal{P}^{m-\gamma(s)}$. From this we will deduce that $y\mathcal{P}^{\delta(m)} \neq 0$. This proves 1:3 since y is arbitrary.

Pick $K \geq 0$ such that $m \equiv \gamma(K) \bmod p^{K+1}$. By 4:10 $K > 0$. From 4:5 and 4:9 we can deduce

LEMMA 4:12. (i) *$y\mathcal{P}^q = 0$ unless $q \equiv 0 \bmod p^K$;*
(ii) *$y\mathcal{P}^q = 0$ if $q > \delta(m)$.*

We consider how $\mathcal{P}^{m-\gamma(s)}$ acts on $\bigotimes_{i=1}^p y$ in three cases:

(i) $1 < s < K$: Since $m - \gamma(s) \not\equiv 0 \bmod p^K$ it follows, from 4:12(i), and the Cartan formula that $\bigotimes_{i=1}^p y \in \ker \mathcal{P}^{m-\gamma(s)}$.

(ii) $s = K$: Since $m - \gamma(K) = p\delta(m)$ it follows, from 4:12(ii), and the Cartan formula that $(\bigotimes_{i=1}^p y)\mathcal{P}^{m-\gamma(K)} = \bigotimes_{i=1}^p y\mathcal{P}^{\delta(m)}$. Hence, we can assume $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(K)}$ since, otherwise, we are done.

(iii) $s > K$: By (i) and (ii) there exists $s > K$ such that $(\bigotimes_{i=1}^{p} y)P^{m-\gamma(s)} \neq 0$.

(a) We show $yP^{p^K} \neq 0$. Now, by the definition of K, $m - \gamma(s) \equiv (p-1)p^K \bmod p^{K+1}$. Hence, by the Cartan formula and 4:12(i), we can find q where $q \equiv \alpha p^K \bmod p^{K+1}$ $(1 \leqslant \alpha \leqslant p-1)$ such that $yP^q \neq 0$. By 4:6 and 4:12(i), we conclude

$$yP^{p^K}P^{q-p^K} = \beta y P^q \neq 0$$

where $\beta \in Z_p$. Hence $yP^{p^K} \neq 0$.

(b) We show $yP^{\delta(m)} \neq 0$. Let $n = m - p^{K+1} + p^K$. By the induction hypothesis $yP^{p^K}P^{\delta(n)} \neq 0$. Now $\delta(m) = \delta(n) + p^K$ and $\delta(n) \equiv 0 \bmod p^K$. For, since $m \equiv \gamma(K) \bmod p^{K+1}$ it follows, from 4:5, that $m \equiv p^{K+1} + \gamma(K) \bmod p^{K+2}$ and, thus, $n \equiv \gamma(K+1) \bmod p^{K+2}$. From 4:6 and 4:12(i) we then deduce that

$$yP^{p^K}P^{\delta(n)} = \lambda y P^{\delta(n)+p^K} = \lambda y P^{\delta(m)}$$

where $\lambda \in Z_p$. Hence $yP^{\delta(m)} \neq 0$. Q.E.D.

5. **Proof of Theorem 1:1.** In this section we will prove Theorem 1:1. We will prove it by increasing induction on the dimension. For $m = 1$ the theorem is true. Assume that $Q(H^{2i}) \neq 0$ only for binary i if $i < m$. By the proofs given in the last section (which were also by induction on the dimension) we can assume 1:2 and 1:3 hold in dimensions less than $2m$. Assume $Q(H^{2m}) \neq 0$ and m is not binary. We will produce a contradiction thereby showing these two conditions are incompatible.

Let k be the minimal $s \geqslant 0$ such that P^{p^s} acts nontrivially on $P(H_{2m})$. Let $n = m - p^{k+1} + p^k$. By the induction hypothesis n is binary. Although m is not binary we define $\delta(m) = \delta(n) + p^k$. Let t be the maximal $s \geqslant 0$ such that $m_s \neq 0$ in the p-adic expansion $\Sigma m_s p^s$ of m.

Pick $0 \neq y \in P(H_{2m})$. We will prove three propositions which, together, clearly produce the desired contradiction.

PROPOSITION 5:1. $y^p = 0$.

PROPOSITION 5:2. *$\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(s)}$ for $s > 0$ unless $s = t$ and $m - \gamma(t) = (p-1)\delta(m)$. And, in that case $\bigotimes_{i=1}^{p} y \in \ker P^{(p-1)\delta(m)}P^{\delta(m)}$.*

PROPOSITION 5:3. *If $\bigotimes_{i=1}^{p} y \in \bigcap_{0<s<t} \ker P^{m-\gamma(s)}$ and $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(t)}P^{\delta(m)}$ where $m - \gamma(t) = (p-1)\delta(m)$, then $y^p \neq 0$.*

Before proving these propositions we establish some technical results.

LEMMA 5:4. *$\mathcal{P}^q$ acts trivially on $P(H_{2m})$ unless $m \equiv 0 \bmod p^k$.*

PROOF. This follows from the definition of k and 4:6. We work by induction as in 4:8. Q.E.D.

Let $m = \Sigma m_s p^s$ and $n = \Sigma n_s p^s$ be the p-adic expansions of m and n respectively.

PROPOSITION 5:5. $m \equiv 2p^{k+1} + \gamma(k) \bmod p^{k+2}$. *Furthermore, for $s \geq k + 2$, $m_s = n_s$.*

PROOF. It suffices to show $n \equiv \gamma(k+1) \bmod p^{k+1}$. For then, in order for n to be binary and m not binary, 5:5 is the only possibility. Hence we can assume $k \geq 1$ since, by 1:3, $n \equiv 1 \bmod p$. We break our proof into two stages:

(a) We show $\delta(n) \equiv 0 \bmod p^{k-1}$. Pick $0 \neq y \in P(H_{2m})$. Then, since $m \equiv n \equiv 1 \bmod p$, it follows as in 4:11 that $y^p = 0$. By 4:3, $\bigotimes_{i=1}^{p} y \notin \bigcap_{1 \leq s} \ker \mathcal{P}^{m-\gamma(s)}$. Hence, by 5:4 and the Cartan formula, there exists $s \geq 1$ such that $m - \gamma(s) \equiv 0 \bmod p^k$. Hence $n \equiv m \equiv \gamma(s) \bmod p^k$ and $\delta(n) \equiv 0 \bmod p^{k-1}$.

(b) We show $\rho(n) \geq \gamma(k+1)$. Pick $y \in P(H_{2m})$ such that $y\mathcal{P}^{p^k} \neq 0$. By 1:3 $y\mathcal{P}^{p^k}\mathcal{P}^{\delta(n)} \neq 0$. Since $\delta(n) \equiv 0 \bmod p^{k-1}$ it follows, from 5:4 and 4:6, that

$$(*) \qquad y\mathcal{P}^{p^k}\mathcal{P}^{(n)} = \alpha y\mathcal{P}^{\delta(n)+p^k} + \beta y\mathcal{P}^{\delta(n)+p^k-p^{k-1}}\mathcal{P}^{p^{k-1}}$$

where $\alpha, \beta \in Z_p$. We wish to eliminate the possibility that $y\mathcal{P}^{\delta(n)+p^k-p^{k-1}}\mathcal{P}^{p^{k-1}} \neq 0$. First, $y\mathcal{P}^{\delta(n)+p^k-p^{k-1}} \neq 0$ implies, by 5:4, that $\delta(n) \equiv p^{k-1} \bmod p^k$. Then, secondly, when we dualize, $\mathcal{P}^{p^{k-1}}$ acting nontrivially on $Q(H^{2\delta(n)+2\rho(n)})$ implies, by 4:5, that $\delta(n) \equiv p^{k-1} \bmod p^{k+1}$. Thus $n \equiv p^k + \gamma(s) \bmod p^{k+2}$ where $s < k$. But then, thirdly, n binary implies m binary which is not true. Thus $(*)$ reduces to

$$(**) \qquad y\mathcal{P}^{p^k}\mathcal{P}^{\delta(n)} = \alpha y\mathcal{P}^{\delta(n)+p^k} \neq 0.$$

But $\mathcal{P}^{\delta(n)+p^k}$ acting nontrivially on $Q(H^{2\delta(n)+2\rho(n)})$ implies $\rho(n) > p^k$. Hence $\rho(n) \geq \gamma(k+1)$. Q.E.D.

(I) PROOF OF PROPOSITION 5:1. Let [,] be the Lie bracket product.

LEMMA 5:6. *If $u, v \in H_*$ then $[u, v]\mathcal{P}^q = \Sigma_{i+j=q}[u\mathcal{P}^i, v\mathcal{P}^j]$.*

The proof of 5:6 is straightforward.

LEMMA 5:7. *If $u \in P(H_{2i})$, $v \in P(H_{2j})$, $i, j < m$, then $[u, v] = 0$.*

PROOF. By 1:3, $i \equiv j \equiv 1 \bmod p$. Hence $u\mathcal{P}^1 = v\mathcal{P}^1 = 0$. By 5:6, $[u, v]\,\mathcal{P}^1 = 0$. But u, v primitive implies that $[u, v] \in P(H_{2i+2j})$. Since $i + j \equiv 2 \bmod p$ it follows by 4:4 that $[u, v] = 0$. Q.E.D.

LEMMA 5:8. *If $u \in P(H_{2i})$, $v \in P(H_{2m})$ and $i < m$ then* $[u, v] = 0$.

PROOF. By contradiction. Pick the minimal i such that $[u, v] \neq 0$ where $u \in P(H_{2i})$ and $v \in P(H_{2m})$. Then $[u, v] \in P(H_{2m+2i})$. But, by 5:6, 5:7, and our choice of i, $[u, v]\,\mathcal{P}^q = 0$ if $q > 0$. Hence, by 4:4, $[u, v] \neq 0$ implies $m + i = \gamma(s)$ for some $s \geqslant 1$. But by 5:5 and the fact that i is binary this is impossible. Q.E.D.

We now prove 5:1. By 4:4 it suffices to prove $y^p\mathcal{P}^1 = 0$ since $y^p \in P(H_{2pm})$. And, by 5:8, $y^p\mathcal{P}^1 = p(y\mathcal{P}^1)y^{p-1} = 0$. Q.E.D.

(II) PROOF OF PROPOSITION 5:2. We begin by seeing how $\mathcal{P}^q$ acts on $y \in P(H_{2m})$.

LEMMA 5:9. $y\mathcal{P}^q = 0$ *unless* $q \not\equiv 0 \bmod p^{k+1}$.

PROOF. By contradiction. We suppose that $y\mathcal{P}^q \neq 0$ and that $q \equiv 0 \bmod p^{k+1}$ for $q > 0$.

(a) We show $y\mathcal{P}^{p^{k+1}} \neq 0$. By 5:5 and the fact that 1:1 is true in dimensions less than $2m$, we must have $q \not\equiv 0 \bmod p^{k+2}$. By 4:6 and 5:4 we have

$$y\mathcal{P}^q = \alpha y\mathcal{P}^{p^{k+1}}\mathcal{P}^{q-p^{k+1}} + \beta y\mathcal{P}^{q-p^k}\mathcal{P}^{p^k}$$

where $\alpha, \beta \in Z_p$. But $q - p^k \equiv (p-1)p^k \bmod p^{k+1}$. Therefore, by 5:5 and the fact that 1:1 is true in dimensions less than m, $y\mathcal{P}^{q-p^k} = 0$. Hence, $y\mathcal{P}^{p^{k+1}}\mathcal{P}^{q-p^{k+1}} \neq 0$.

(b) By 5:5 and the fact that 1:1 is true in dimensions less than $2m$, $y\mathcal{P}^{p^{k+1}} \neq 0$ produces an obvious contradiction unless $p = 3$. We now give an argument which shows there is a contradiction for $p = 3$ as well. Let $l = m - 2 \cdot 3^{k+1}$. By 5:5, $\delta(l) \equiv 0 \bmod 3^{k+1}$ and $\rho(l) = \gamma(k) < 3^{k+1}$. By 1:3, $y\mathcal{P}^{3^{k+1}} \neq 0$ implies $y\mathcal{P}^{3^{k+1}}\mathcal{P}^{\delta(l)} \neq 0$. By an argument similar to that in (a)

$$y\mathcal{P}^{\delta(l)+3^{k+1}} = \lambda y\mathcal{P}^{3^{k+1}}\mathcal{P}^{\delta(l)} \neq 0$$

where $\lambda \in Z_p$. Dualizing, we have $\mathcal{P}^{\delta(l)+3^{k+1}}$ acting nontrivially on $Q(H^{2\delta(l)+2\rho(l)})$. Since $\rho(l) < 3^{k+1}$ this is a contradiction. Q.E.D.

LEMMA 5:10. *If $y\mathcal{P}^q \neq 0$ then $q \equiv p^k \bmod p^{k+2}$. Also q is binary and $\delta(m) - q$ is binary. In particular, $q \leqslant \delta(m)$.*

PROOF. The first statement follows from 5:5, 5:9 and the fact that 1:1 is true in dimensions less than $2m$. Regarding the second statement, we can deduce from 4:6, 5:4 and the first statement that

$$yP^{p^k}P^{q-p^k} = \alpha y P^q \neq 0$$

($\alpha \in Z_p$). Then, since P^{q-p^k} acts nontrivially on $P(H_{2n})$, we can deduce, from 4:9 and 4:5, that $q - p^k$ and $\delta(n) - (q - p^k)$ are binary. Hence q and $\delta(m) - q = \delta(n) + p^k - q = \delta(n) - (q - p^k)$ are binary. Q.E.D.

This is one more technical lemma we will have need of.

LEMMA 5:11. *For $s < t$, $m - \gamma(s) > (p - 1)\delta(m)$.*

PROOF. Since $m = n + (p - 1)p^k$ and $(p - 1)\delta(m) = (p - 1)(\delta(n) + p^k) = (p - 1)\delta(n) + (p - 1)p^k$, it suffices to show $n - \gamma(s) > (p - 1)\delta(n)$. Now $n = p\delta(n) + \rho(n)$. Hence $n - (\delta(n) + \rho(n)) = (p - 1)\delta(n)$. Therefore it suffices to show $\delta(n) + \rho(n) > \gamma(s)$ for $s < t$. And this is obvious. Q.E.D.

We now prove 5:2. The first part of the proof consists of determining when $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(s)}$. Given s it suffices to show $\bigotimes_{i=1}^{p} y P^{q_i} = 0$ whenever $\Sigma_{i=1}^{q} q_i = m - \gamma(s)$. We consider s in four cases.

(i) $0 < s < k$. Since $\Sigma q_i = m - \gamma(s) \not\equiv 0 \bmod p^k$ it follows by 5:4 that $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(s)}$.

(ii) $s = k, k + 1$. Since $\Sigma q_i = m - \gamma(s) \equiv 2p^{k+1}$ or $p^{k+1} + (p - 1)p^k \bmod p^{k+2}$ it follows from 5:10 that $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(s)}$.

(iii) $k + 1 \leqslant s < t$. Since $\Sigma q_i = m - \gamma(s) \equiv (p - 1)p^k \bmod p^{k+2}$ we must have, by 5:10, $q_i = 0$ for some i and $\Sigma q_i \leqslant (p - 1)\delta(m)$ if $\bigotimes_{i=1}^{p} y P^{m-\gamma(s)} \neq 0$. By 5:11 $\bigotimes_{i=1}^{p} y \in \ker P^{m-\gamma(s)}$.

(iv) $s = t$. Suppose $\bigotimes_{i=1}^{p} y \notin \ker P^{m-\gamma(t)}$. We will show $m - \gamma(t) = (p - 1)\delta(m)$.

Let $\Sigma a_s p^s$ and $\Sigma b_s p^s$ be the p-adic expansion of $m - \gamma(t)$ and $(p - 1)\delta(m)$ respectively. By arguing as in (iii) we see that $q_i = 0$ for some i. Also q_i and $\delta(m) - q_i$ are binary for each i. Since $\Sigma q_i = m - \gamma(t)$ it follows that

$$(*) \qquad a_s \neq 0 \text{ only if } b_s \neq 0.$$

But if $\Sigma m_s p^s$ is the p-adic expansion of m it follows from the definition of $\delta(m)$ that $b_s \neq 0$ only if $m_{s+1} \neq 0$. Hence

$$(**) \qquad a_s \neq 0 \text{ only if } m_{s+1} \neq 0.$$

Let r be the minimum $s \geqslant k + 2$ such that $m_s = 0$. Then $r < t$ since otherwise $m - \gamma(t) = (p - 1)\delta(m) = (p - 1)p^k$ and we are done.

But it is easy to see

$$(***) \qquad a_s \neq 0 \quad \text{for } r \leqslant s < t.$$

It follows that $m_s \neq 0$ for $r < s \leqslant t$. From this we deduce that

$$m - \gamma(t) = (p-1)\delta(m) = (p-1)p^k + \sum_{r<s<t} (p-1)p^s.$$

The second and final part of the proof of 5:2 consists of showing that $\bigotimes_{i=1}^{p} y \in \ker P^{(p-1)\delta(m)}P^{\delta(m)}$ when $m - \gamma(t) = (p-1)\delta(m)$. Let $z = yP^{\delta(m)}$. Since the dimension of z is $m - (p-1)\delta(m) = \gamma(t)$ it follows from 1:2 that $zP^i = 0$ for $i > 0$. Now, by 5:10,

$$\left(\bigotimes_{i=1}^{p} y\right) P^{(p-1)\delta(m)} = \sum_{i=1}^{p} z \otimes z \otimes \cdots \otimes z \otimes y \otimes z \otimes \cdots \otimes z$$

ith place.

Hence

$$\left(\bigotimes_{i=1}^{p} y\right) P^{(p-1)\delta(m)} P^{\delta(m)} = p\left(\bigotimes_{i=1}^{p} z\right) = 0. \quad \text{Q.E.D.}$$

(III) PROOF OF PROPOSITION 5:3.

LEMMA 5:12. $Q(H^{2m}) = P^{\delta(m)} Q(H^{2\delta(n)+2\rho(n)})$.

PROOF. Pick $0 \neq y \in P(H_{2m})$. By 4:4 there exists $q > 0$ such that $yP^q \neq 0$. By 5:10 $q \equiv p^k \bmod p^{k+1}$. By 4:6 and 5:4 $yP^{p^k}P^{q-p^k} = \alpha y P^q \neq 0$. Hence $yP^{p^k} \neq 0$. By 1:3, $yP^{p^k}P^{\delta(n)} \neq 0$. By 5:5, $\delta(n) \equiv 0 \bmod p^k$. Again, using 4:6 and 5:4,

$$yP^{\delta(m)} = yP^{\delta(n)+p^k} = \alpha y P^{p^k} P^{\delta(n)} \neq 0. \quad \text{Q.E.D.}$$

LEMMA 5:13. $\delta(n) + \rho(n) = m - (p-1)\delta(m) = \gamma(t)$.

This follows from the identities $m - \gamma(t) = (p-1)\delta(m)$ and

$$m = n + (p-1)p^k = p\delta(n) + \rho(n) + (p-1)p^k$$

$$= \delta(n) + \rho(n) + (p-1)\delta(m). \quad \text{Q.E.D.}$$

From 4:1, 4:2 and the definition of t we can deduce the Adem relation

$$(5.14) \quad Q_0 P^m P^{\delta(m)} = \sum_{0<s<t} (-1)^s P^{m-\gamma(s)} Q_s P^{\delta(m)} + (-1)^t P^{m-\gamma(t)} P^{\delta(m)} Q_t.$$

Our proof of 5:3 is a variation of that for 4:3. We use the notation of that proof. Expand y to a basis of $P(H_{2m})$ and pick $x \in Q(H^{2m})$ dual to y. By 5:12 and 5:13 there exists $z \in Q(H^{\gamma(t)})$ such that $x = P^{\delta(m)}(z)$. Now z is represented by $z \in H^{\gamma(t)}$ such that $\bar{\mu}^*(z) \in F_{\gamma(t)-1} \otimes F_{\gamma(t)-1}$. Let ϕ be the unstable secondary operation corresponding to 5:14 where $\theta = P^{\delta(m)}$, $a_s = (-1)^s P^{m-\gamma(s)}$ and

$b_s = Q_s P^{\delta(m)}$ if $s < t$ while $a_t = (-1)^t P^{m-\gamma(t)} p^{\delta(m)}$ and $b_t = Q_t$. Then ϕ is defined on z. The rest of the proof follows as in 4:3. There is one extra complication which arises however. Since $|\theta| = |P^{\delta(m)}| > 0$ we have to ensure that the difficulty discussed after 3:4 does not arise. That is, we have to show that $x \notin Q(F_{\gamma(t)-1})$. But, as was shown in the proof of 5:10, if $yP^q \neq 0$ then $yP^q = \alpha y P^{p^k} P^{q-p^k}$ $(\alpha \in Z_p)$. Thus P^q acting on y factors through $P(H_{2n})$. By 1:2 it then follows that $x \notin Q(F_{\gamma(t)-1})$. Q.E.D.

REFERENCES

1. W. Browder, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51. MR **23** #A2201.
2. ———, *On differential Hopf algebras*, Trans. Amer. Math. Soc. **108** (1963), 153–176. MR **26** #3061.
3. J. Harper and C. Schochet, *Coalgebra extensions in two stage Postinikov systems*, Math. Scand. **29** (1971), 232–236. MR **46** #9985.
4. J. Harper, *On the cohomology of stable two stage Postnikov systems*, Trans. Amer. Math. Soc. **152** (1970), 375–388. MR **42** #3789.
5. R. Kane, *On loop spaces without p torsion*, Pacific J. Math. **60** (1975), 189–201.
6. ———, *Primitivity and finite H-spaces*, Oxford Quart. **26** (1975), 309–313.
7. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171. MR **20** #6092.
8. J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR **30** #4259.
9. A. Zabrodsky, *Secondary operations in the cohomology of H-spaces*, Illinois J. Math. **15** (1971), 648–655. MR **48** #1234.
10. ———, *Secondary cohomology operations in the module of indecomposables*, Algebraic Topology Conference, Aarhus, 1970.

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